Quantum Introductory Tutorial (3h)

DID YOU KNOW THAT A QUBIT IS A...   NO SPOILER!

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Spring School in Theoretical Computer Science (EPIT)
on Quantum Computer Science

May 24, 2021

Bring pen & paper ;-)
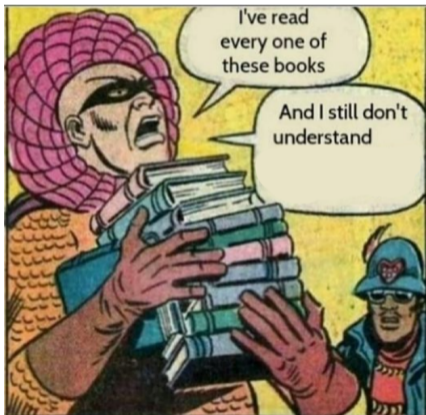
References

https://www.humanatease.com/entries/links/142857

- *Introduction to Quantum Computing* (2005), John Watrous.
  (we will cover chapters 1-4)
- *Quantum Computation and Quantum Information* (2010), Michael A. Nielsen, Isaac L. Chuang.
- *ZX-calculus for the working quantum computer scientist* (2020), J. van de Wetering.
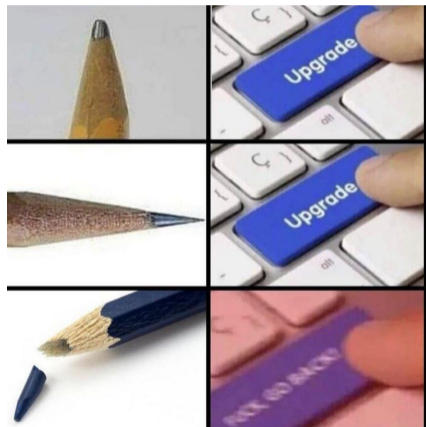
Goal of this tutorial

Outline

- First intuition behind quantum information
- Mathematical (and graphical Yes, drawings are great. Like ZX-Calculus.) formalism
- Applications:
  - Superdense coding
  - Quantum teleportation
  - Deutsch's algorithm

Practice

Goal: **YOU** practice $\Rightarrow$ Polls (small exercices) + Breakrooms (longer exercices at the end).
**Bring pen & paper**.

https://gr.pinterest.com/pin/734227545481401724/

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Poll

- A = Quant... what?
- B = Don't know/remember formal definition of qubits/measurements/tensors...
- C = Don't know much about quantum algorithms/protocols
- D = Expert in quantum computing, don't know ZX-Calculus
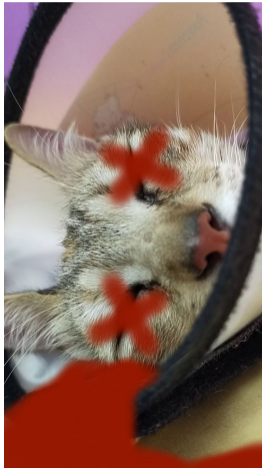- E = Expert in quantum computing, can use ZX-Calculus

Breakout rooms
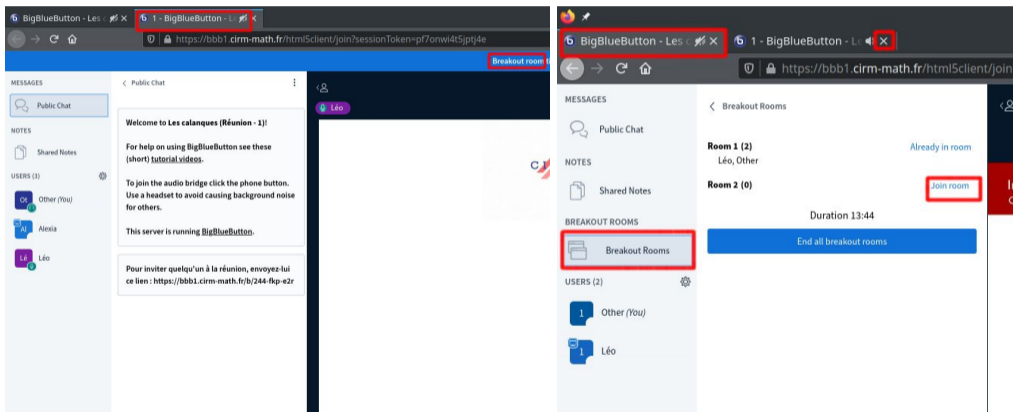
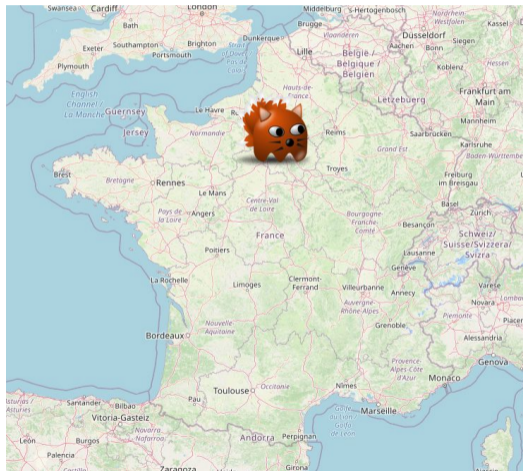# Classically: superposition is impossible

Classically: superposition is impossible

No cat was harmed in the making of these slides

Classically: superposition is impossible

Classically: superposition is impossible

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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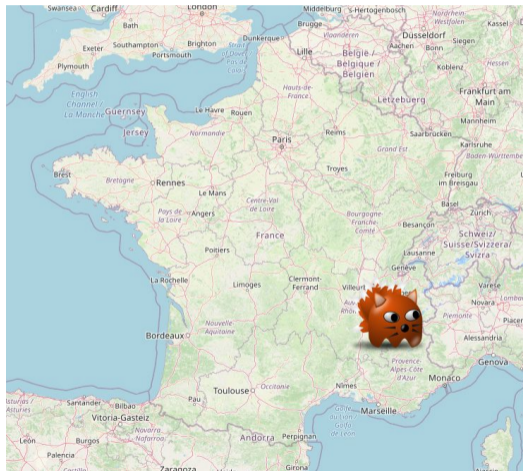
Classically: information can be copied/pasted

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(or M-w/C-y)
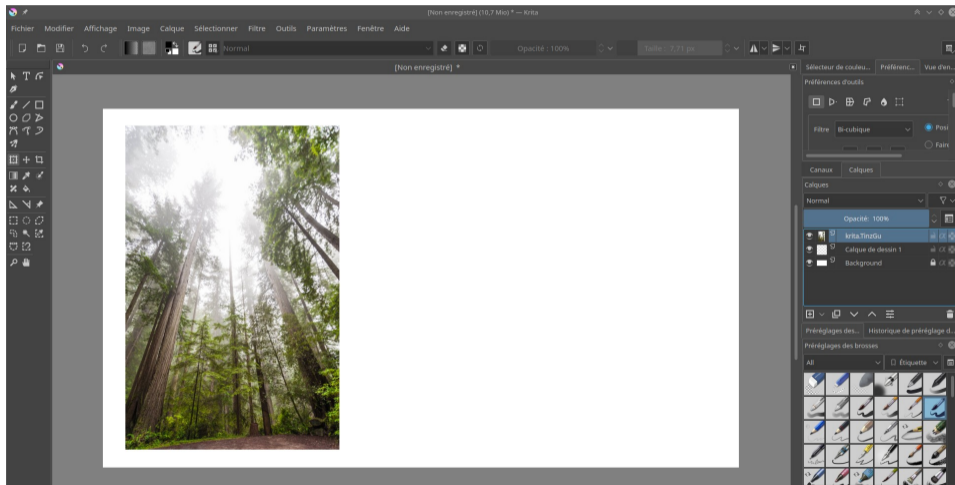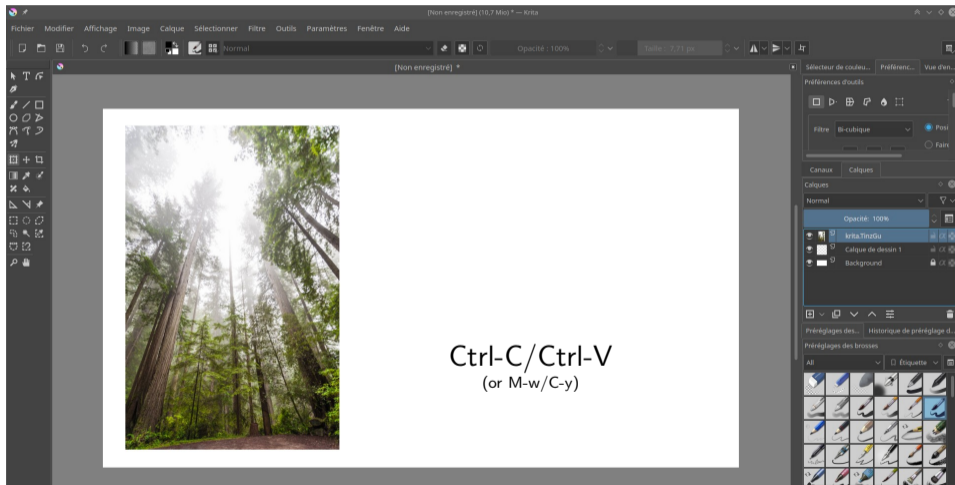
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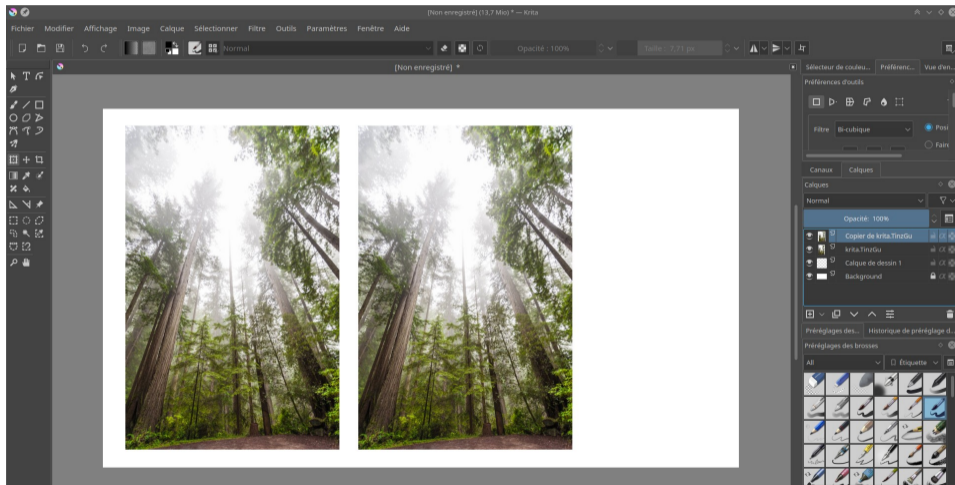
Classically: observation does not alter the object

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Classically: modifications are local and can't propagate faster than light

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Quantum: different laws appear at very small scales
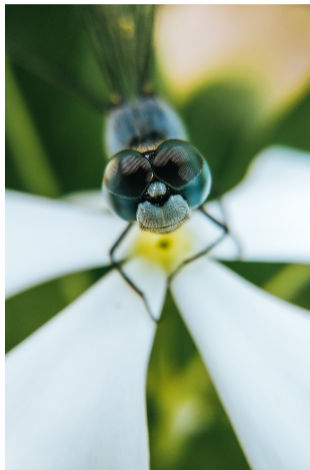
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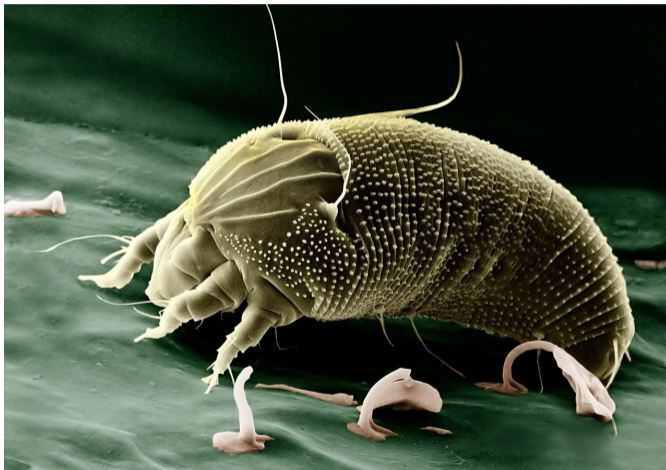
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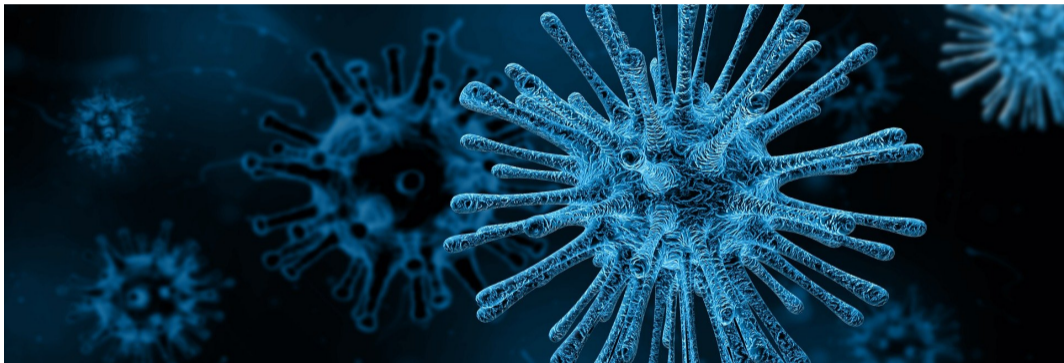
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Quantum laws: superposition is possible

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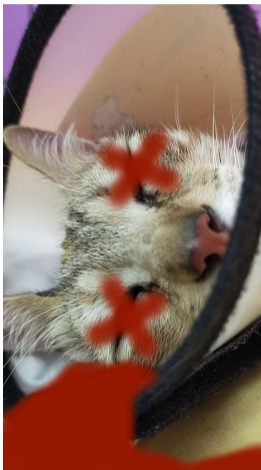
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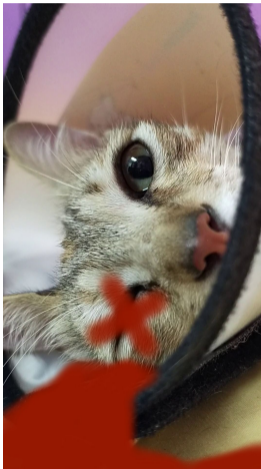
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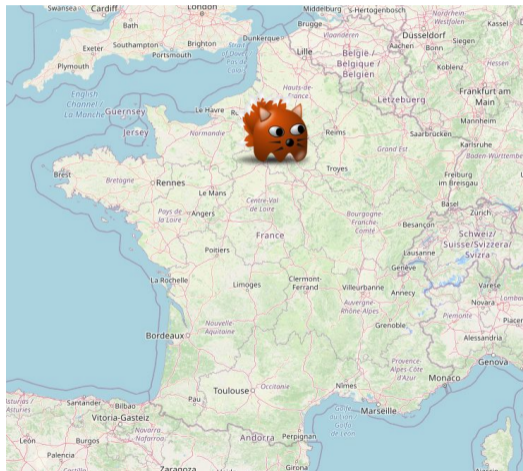
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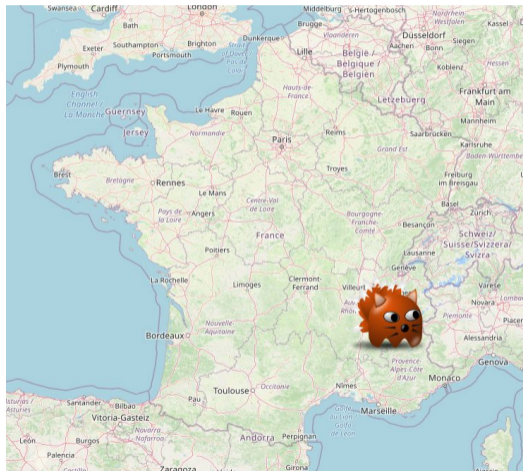
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Quantum laws: superposition is possible

Advantages of superposition

- Computationally: compute in superposition = "**parallelize**" an operation on exponentially many inputs in one step.
  ⇒ **Quantum speedup** (hopefully)
  Example: Shor's algorithm can factor efficiently numbers using this trick (among others).

- Security: hide (see later) multiple values in one object

Quantum laws: observation (measurement) alters the state

Quantum laws: observation (measurement) alters the state

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Quantum laws: observation (measurement) alters the state

Holevo's theorem (over-simplified)

From a quantum state of dimension 2^n, it is impossible to extract more than n bits of information.

No-cloning theorem (simplified)

It is impossible to clone an arbitrary quantum state in multiple exact copies.

Quantum laws: observation (measurement) alters the state

Advantages of having only destructive measurements

Security: It is possible to hide information inside a quantum state.
⇒ **Quantum Key Distribution (QKD)**

Drawbacks of having only destructive measurements

Computationnally: cannot recover all information contained in the superposition.
⇒ **Need to choose the measurement cleverly**
Often need to repeat multiple times:

- either because the (probabilist) measurement failed to give the wanted result
- or because the measurement does not give enough information (repeat + classical post-processing)
- or both!

Quantum laws: observation (measurement) alters the state
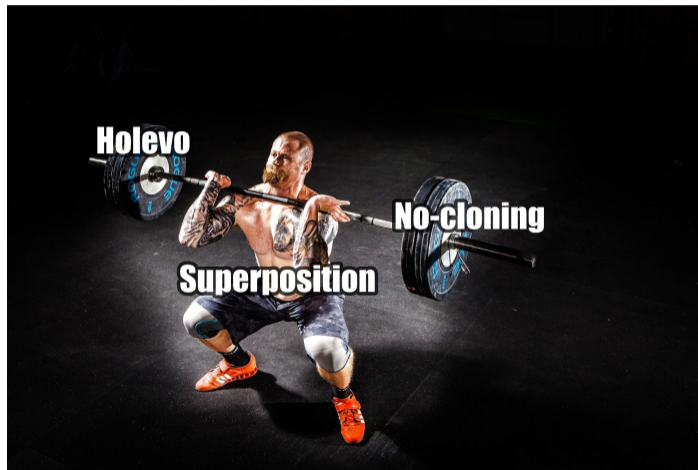
Quantumly: modifications are non-local

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Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Quantumly: modifications are non-local

Advantages of non-local modifications

Entangled states can be used to produce **non-local** modifications propagating faster than light

⇒ Possible to obtain new kinds of correlations without communication that are impossible to obtain classically.

Example: Bell's inequality violation can prove that our world is non-local, Magic Square Game, Quantum Teleportation.

Limitations

WARNING: even if modifications can propagate faster than light, quantum laws does **not** allow faster than light communication.

What is (formally) a qubit?

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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What is (formally) a qubit?

A vector!

Classically:
– can be up

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Classically:
– can be up
– can be down

Classically:
– can be up $(p = 0.3)$
– can be down $(p = 0.7)$

Don't know state.
Know distribution.

Classically:
– can be up ($p = 0.3$)
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$$\mathbf{v} = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} \in [0, 1]^2$$

Norm: $0.3 + 0.7 = 1$

Classically:
– can be up ($p = 0.3$)
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Don't know state.
Know distribution.

Apply operations:
– Rotation

$$\mathbf{v} = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} \in [0,1]^2$$

Norm: $0.3 + 0.7 = 1$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}$$

$$= \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$$

Classically:
– can be up ($p = 0.3$)
– can be down ($p = 0.7$)

Don't know state.
Know distribution.

Apply operations:
– Rotation
– Measurement

$$\mathbf{v} = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} \in [0,1]^2$$

Norm: $0.3 + 0.7 = 1$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}$$

$$= \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Classically

- Only two actual states (up or down), $\mathbf{v}$ is only our knowledge about the state

- $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \in [0,1]^2$, and the normalization is $a + b = 1$.

Quantumly:
$\mathbf{v}$ is complex
$\mathbf{v}$ **is** the actual state.

– can be up

$$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2$$

Norm: $|a|^2 + |b|^2 = 1$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Quantumly:
v is complex
v **is** the actual state.

– can be up
– can be down
– can be in superposition

$$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix} \in \mathbb{C}^2$$

Norm: $|a|^2 + |b|^2 = 1$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Quantumly:
$\mathbf{v}$ is complex
$\mathbf{v}$ **is** the actual state.

– can be up
– can be down
– can be in superposition (but no full access)

$$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$$

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Quantumly:
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– can be down
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Apply operations:
– Unitaries: $U^\dagger U = I$
($U^\dagger = \bar{U}^T$)

$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$

$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix} \in \mathbb{C}^2$

Norm: $|a|^2 + |b|^2 = 1$

$\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix}$

$= \begin{pmatrix} \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Quantumly:
v is complex
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– can be up
– can be down
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Apply operations:
– Unitaries: $U^\dagger U = I$
$(U^\dagger = \bar{U}^T)$
– Measurements

$$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix} \in \mathbb{C}^2$$

Norm: $|a|^2 + |b|^2 = 1$

$$\Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Classically

- Only two actual states (up or down), $\mathbf{v}$ is only our knowledge about the state
- $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \in [0,1]^2$, and the normalization is $a + b = 1$.

Quantumly

- $\mathbf{v}$ **is** the actual "qubit" state (many possible values)
- $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$ (Hilbert Space $\mathcal{H}_2$), and the normalization is $|a|^2 + |b|^2 = 1$.
- Possible operations are unitaries and measurements (see later)

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Unitaries

First kind of allowed operations: unitaries

U must preserve Euclidean norm in Hilbert Space, i.e. unitary: $U^\dagger U = I$, always invertible
($U^\dagger := \bar{U}^T$ complex conjugate + transpose)

Example "dagger" $A^\dagger$ (not unitary here)

$$\begin{pmatrix} e^{i\pi/4} & 2i \\ 1+i & 4 \end{pmatrix}^\dagger = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

Poll: value of a_3 ? A=$1+i$, B=$1-i$, C=$2i$, D=$-2i$, E=Other.

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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($U^\dagger := \bar{U}^T$ complex conjugate + transpose)

Example "dagger" $A^\dagger$ (not unitary here)

$$\begin{pmatrix} e^{i\pi/4} & 2i \\ 1+i & 4 \end{pmatrix}^\dagger = \begin{pmatrix} e^{-i\pi/4} & 1-i \\ -2i & 4 \end{pmatrix}$$

Poll: answer D$=-2i$.

Unitaries

First kind of allowed operations: unitaries

U must preserve Euclidean norm in Hilbert Space, i.e. unitary: $U^\dagger U = I$, always invertible
($U^\dagger := \bar{U}^T$ complex conjugate + transpose)

Exercice

Poll: Is this matrix a unitary matrix?

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Introduction
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Formalisation: single qubit
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Multiple qubits
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$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

⇒ True!

Unitaries

First kind of allowed operations: unitaries

U must preserve Euclidean norm in Hilbert Space, i.e. unitary: $U^\dagger U = I$, always invertible

($U^\dagger := \bar{U}^T$ complex conjugate + transpose)

Example

- $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I^\dagger = I$, $I^\dagger I = I$

- $H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$, $H^\dagger = H$, $H^\dagger H = I$

- $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $X^\dagger = X$, $X^\dagger X = I$

- $R_Z(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, $Z := R_Z(\pi)$, $R_Z(\theta)^\dagger = R_Z(-\theta)$, $R_Z(\theta)^\dagger R_Z(\theta) = I$.

Unitaries

Hadamard gate

$$H \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$H \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Useful gate to create a uniform superposition, and to change basis ($H^2 = I$).

Introduction
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Circuit/ZX
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Applications/TD
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Measurements

Second kind of allowed operations: measurements

If we get a qubit $\begin{pmatrix} a \\ b \end{pmatrix}$ and measure it, we get one (classical) bit:

- outcome 0 with probability $|a|^2$. Qubit changed into $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

- outcome 1 with probability $|b|^2$. Qubit changed into $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Properties

A measurement is **destructive** (not invertible) and **probabilistic**. Measuring twice gives you the same outcome twice.

Measurements

Different kinds measurements

Here: measurement in $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ / $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (computational) basis.

Later: generalize to other basis. Equivalent to unitary + measurement in computational basis.

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Exercice 1

Let $A_1 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $A_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. For which i is it physical to transform any quantum state v into $A_i v$?

Exercice 2

Which state(s) are valid quantum states: $v_1 = \begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1/\sqrt{6} \\ \sqrt{5/6} \end{pmatrix}$?

For valid states: probability outcome 0 when measuring in computational basis?

Exercice 3

Let $v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $v_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Given a state $v \in \{v_1, v_2\}$, how can we distinguish if $v = v_1$ or $v = v_2$?

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Exercice 1

Let $A_1 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $A_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. For which[a] i is it physical to transform any quantum state v into $A_i v$?

———————————————————————

[a]**Poll**: For how many...: A=0, B=1, C=2, D=The answer D, E=No idea.

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Exercice 1

Let $A_1 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $A_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. For which[a] i is it physical to transform any quantum state v into $A_i v$?

[a]**Poll**: For how many...: A=0, B=1, C=2, D=The answer D, E=No idea.

Answer: the correct answer is B

Exercice 1

Let $A_1 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $A_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. For which[a] i is it physical to transform any quantum state v into $A_i v$?

[a]**Poll**: For how many...: A=0, B=1, C=2, D=The answer D, E=No idea.

Answer: the correct answer is B

- $A_1^\dagger A_1 = \bar{A_1}^T A_1 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \neq I$. No.

Exercice 1

Let $A_1 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $A_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. For which[a] i is it physical to transform any quantum state v into $A_i v$?

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[a]**Poll**: For how many...: A=0, B=1, C=2, D=The answer D, E=No idea.

Answer: the correct answer is B

- $A_1^\dagger A_1 = \bar{A_1}^T A_1 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \neq I.$ No.

- $A_2^\dagger A_2 = \bar{A_2}^T A_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$ Yes.

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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Exercice 1

Let $A_1 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $A_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. For which[a] i is it physical to transform any quantum state v into $A_i v$?

---

[a]**Poll**: For how many...: A=0, B=1, C=2, D=The answer D, E=No idea.

Answer: the correct answer is B

- $A_1^\dagger A_1 = \bar{A_1}^T A_1 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \neq I$. No.

- $A_2^\dagger A_2 = \bar{A_2}^T A_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. Yes.

- $A_3^\dagger A_3 = \bar{A_3}^T A_3 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq I$. No.

Exercice 2

Which[a] state(s) are valid quantum states: $v_1 = \begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1/\sqrt{6} \\ \sqrt{5/6} \end{pmatrix}$?

For valid states: probability outcome 0 when measuring in computational basis?

---

[a]**Poll**: A=None, B=v_1, C=v_2, D=Both, E=No idea.

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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Exercice 2

Which[a] state(s) are valid quantum states: $v_1 = \begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1/\sqrt{6} \\ \sqrt{5/6} \end{pmatrix}$?

For valid states: probability outcome 0 when measuring in computational basis?

_____

[a]**Poll**: A=None, B=v_1, C=v_2, D=Both, E=No idea.

Answer: correct answer is C

- $\|v_1\|_2^2 = |1/4|^2 + |3/4|^2 = \frac{1}{16} + \frac{9}{16} = \frac{10}{16} \neq 1$. No.

Exercice 2

Which[a] state(s) are valid quantum states: $v_1 = \begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1/\sqrt{6} \\ \sqrt{5/6} \end{pmatrix}$?

For valid states: probability outcome 0 when measuring in computational basis?

[a]**Poll**: A=None, B=v_1, C=v_2, D=Both, E=No idea.

Answer: correct answer is C

- $\|v_1\|_2^2 = |1/4|^2 + |3/4|^2 = \frac{1}{16} + \frac{9}{16} = \frac{10}{16} \neq 1$. No.

- $\|v_2\|_2^2 = \left|\frac{1}{\sqrt{6}}\right|^2 + \left|\sqrt{\frac{5}{6}}\right|^2 = \frac{1}{6} + \frac{5}{6} = 1$. Yes.

Exercice 2

Which[a] state(s) are valid quantum states: $v_1 = \begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1/\sqrt{6} \\ \sqrt{5/6} \end{pmatrix}$?

For valid states: probability outcome 0 when measuring in computational basis?

---
[a]**Poll**: A=None, B=v_1, C=v_2, D=Both, E=No idea.

Answer: correct answer is C

- $\|v_1\|_2^2 = |1/4|^2 + |3/4|^2 = \frac{1}{16} + \frac{9}{16} = \frac{10}{16} \neq 1$. No.

- $\|v_2\|_2^2 = \left|\frac{1}{\sqrt{6}}\right|^2 + \left|\sqrt{\frac{5}{6}}\right|^2 = \frac{1}{6} + \frac{5}{6} = 1$. Yes.

  Probability outcome 0 is $\left|\frac{1}{\sqrt{6}}\right|^2 = \frac{1}{6}$.

  Probability outcome 1 is $\left|\sqrt{\frac{5}{6}}\right|^2 = \frac{5}{6}$.

Exercice 3

Let $v_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Given a state $v \in \{v_0, v_1\}$, how[a] can we distinguish if $v = v_0$ or $v = v_1$?

_____

[a] **Poll**: A=Measure, B=X+Measure, C=Z+Measure, D=H+Measure, E=No idea.

Answer: the correct answer is D

$v_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$: if we measure directly, outcome 0 with probability

$\left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$ and 1 with probability $\frac{1}{2}$... Not useful.

But, if we apply $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ before:

- if $v = v_0$, the state becomes $Hv_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- if $v = v_1$, the state becomes $Hv_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Now, measure the new state: if outcome is 0, we got $v = v_0$, otherwise $v = v_1$.

Dirac Notation

Dirac Notation

Vectors: long to write, especially in high dimensions (see later).

Dirac notation:

- "Ket": $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $|\psi\rangle$: any quantum state,

  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$,

  $|+_\theta\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle) = \begin{pmatrix} 1/\sqrt{2} \\ e^{i\theta}/\sqrt{2} \end{pmatrix}$

- "Bra": $\langle\psi| = |\psi\rangle^\dagger$. Example: $\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $\langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $\langle +_\theta| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{-i\theta} \end{pmatrix}$

Scalar products is "braket": $\langle\psi||\phi\rangle = \langle\psi|\phi\rangle = \overline{\langle\phi|\psi\rangle}$. Projector on $|\psi\rangle$: $|\psi\rangle\langle\psi|$.

Dirac Notation

Exercice

Compute[a] $\langle 0|0 \rangle$ and $\langle 0|1 \rangle$. How to compute probability outcome 0 when measuring $|\psi\rangle$ (computational basis) with Dirac notation?

---
[a]**Poll**: A$=0$ and 0, B$=0$ and 1, C$=1$ and 0, D$=$Other, E$=$No idea.

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Dirac Notation

Exercice

Compute[a] $\langle 0|0\rangle$ and $\langle 0|1\rangle$. How to compute probability outcome 0 when measuring $|\psi\rangle$ (computational basis) with Dirac notation?

---
[a] **Poll**: A=0 and 0, B=0 and 1, C=1 and 0, D=Other, E=No idea.

Answer: correct answer is C

$$\langle 0|0\rangle = \langle 0||0\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \qquad \langle 0|1\rangle = \langle 0||1\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

Not surprising: $|0\rangle$ and $|1\rangle$ is an orthogonal basis. For $(b,c) \in \{0,1\}^2$, $\langle b|c\rangle = \delta_{b,c}$
$\Pr[\text{outcome } = 0] = |\langle 0|\psi\rangle|^2, \qquad \Pr[\text{outcome } = 1] = |\langle 1|\psi\rangle|^2$

Bloch Sphere

Global phase is not observable

Any qubit: $|\psi\rangle = a|0\rangle + b|1\rangle$,
$(a, b) \in \mathbb{C}^2$, $|a|^2 + |b|^2 = 1$ (dimension 4)

$$|\psi\rangle = e^{i\alpha}(c|0\rangle + e^{i\beta}d|1\rangle)$$

$((\alpha, \beta, c, d) \in \mathbb{R}^2$, $c^2 + d^2 = 1)$
Now dimension 3: Bloch Sphere

▶

$|+_\theta\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle)$

Bloch Sphere

Global phase is not observable

Any qubit: $|\psi\rangle = a|0\rangle + b|1\rangle$,
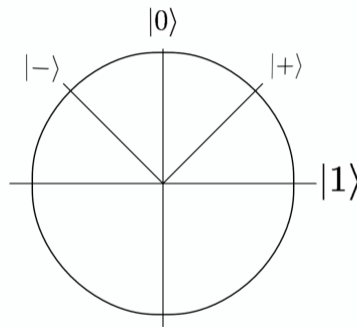$(a,b) \in \mathbb{C}^2$, $|a|^2 + |b|^2 = 1$ (dimension 4)
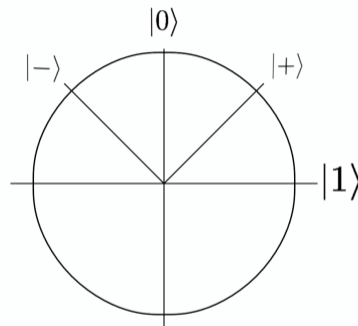
$$|\psi\rangle = \cancel{e^{i\alpha}}(c|0\rangle + e^{i\beta}d|1\rangle)$$

$((\alpha, \beta, c, d) \in \mathbb{R}^2,\ c^2 + d^2 = 1)$
Now dimension 3: Bloch Sphere

▶

$|+_\theta\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle)$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Bloch Sphere: rotations

Rotations on Bloch Sphere

- $R_z(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$

  rotation around Z axis, $Z = R_Z(\pi)$
  Poll: $Z(|+\rangle) =?$ A=$|0\rangle$, B=$|1\rangle$,
  C=$|+\rangle$, D=$|-\rangle$, E=No Idea.

- $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$:

  $(|0\rangle, |1\rangle) \leftrightarrow (|+\rangle, |-\rangle)$

- $R_X(\theta) = H R_Z(\theta) H$, rotation around
  X axis, $X = R_X(\pi)$

- Same idea for Y (less used)

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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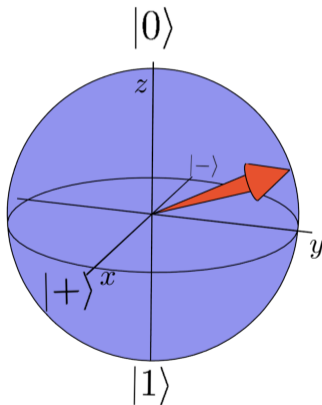## Bloch Sphere: rotations

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Rotations on Bloch Sphere

- $R_z(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$
  rotation around Z axis, $Z = R_Z(\pi)$
  Poll: $Z(|+\rangle) =?$ A=$|0\rangle$, B=$|1\rangle$,
  C=$|+\rangle$, D=$|-\rangle$, E=No Idea. $\Rightarrow |-\rangle$.

- $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$:
  $(|0\rangle, |1\rangle) \leftrightarrow (|+\rangle, |-\rangle)$

- $R_X(\theta) = HR_Z(\theta)H$, rotation around
  X axis, $X = R_X(\pi)$

- Same idea for Y (less used)

Introduction
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Formalisation: single qubit
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Multiple qubits
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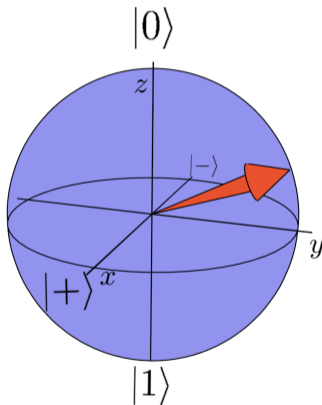
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Measurements on the Bloch Sphere

Visualize measurement on Bloch Sphere

Measurement outcome: proportional to distance (z axis)
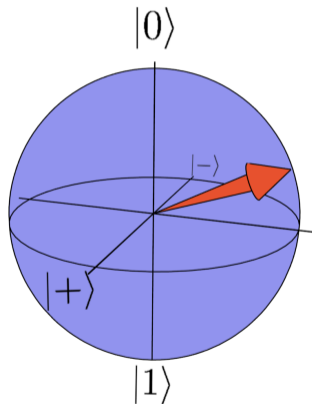
Introduction
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Formalisation: single qubit
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Multiple qubits
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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$$\mathsf{Bat}_1 = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}, \qquad \mathsf{Bat}_2 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$\mathsf{Bat}_1 = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}, \quad \mathsf{Bat}_2 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$\mathsf{Bat}_{1,2} = \begin{pmatrix} 0.3 \times 0.6 \\ 0.3 \times 0.4 \\ 0.7 \times 0.6 \\ 0.7 \times 0.4 \end{pmatrix} \begin{array}{l} \leftarrow \mathsf{Proba}\ (\uparrow, \uparrow) \\ \leftarrow \mathsf{Proba}\ (\uparrow, \downarrow) \\ \leftarrow \mathsf{Proba}\ (\downarrow, \uparrow) \\ \leftarrow \mathsf{Proba}\ (\downarrow, \downarrow) \end{array}$$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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$$\mathsf{Bat}_1 = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}, \quad \mathsf{Bat}_2 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$\mathsf{Bat}_{1,2} = \begin{pmatrix} 0.3 \times 0.6 \\ 0.3 \times 0.4 \\ 0.7 \times 0.6 \\ 0.7 \times 0.4 \end{pmatrix} \begin{matrix} \leftarrow \mathsf{Proba}\ (0,0) \\ \leftarrow \mathsf{Proba}\ (0,1) \\ \leftarrow \mathsf{Proba}\ (1,0) \\ \leftarrow \mathsf{Proba}\ (1,1) \end{matrix}$$
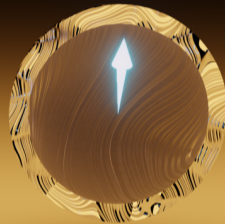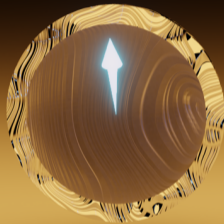
Tensor product
(Kronecker product)

$$\mathsf{Bat}_1 = \begin{pmatrix} {\color{green}0.3} \\ {\color{blue}0.7} \end{pmatrix}, \qquad \mathsf{Bat}_2 = \begin{pmatrix} {\color{orange}0.6} \\ {\color{red}0.4} \end{pmatrix}$$

$$\mathsf{Bat}_{1,2} = \begin{pmatrix} {\color{green}0.3} \times {\color{orange}0.6} \\ {\color{green}0.3} \times {\color{red}0.4} \\ {\color{blue}0.7} \times {\color{orange}0.6} \\ {\color{blue}0.7} \times {\color{red}0.4} \end{pmatrix} \begin{matrix} \leftarrow \mathsf{Proba}\ ({\color{green}0},{\color{orange}0}) \\ \leftarrow \mathsf{Proba}\ ({\color{green}0},{\color{red}1}) \\ \leftarrow \mathsf{Proba}\ ({\color{blue}1},{\color{orange}0}) \\ \leftarrow \mathsf{Proba}\ ({\color{blue}1},{\color{red}1}) \end{matrix} = \begin{pmatrix} {\color{green}0.3} \\ {\color{blue}0.7} \end{pmatrix} \otimes \begin{pmatrix} {\color{orange}0.6} \\ {\color{red}0.4} \end{pmatrix} = \begin{pmatrix} {\color{green}0.3} \times \begin{pmatrix} {\color{orange}0.6} \\ {\color{red}0.4} \end{pmatrix} \\ {\color{blue}0.7} \times \begin{pmatrix} {\color{orange}0.6} \\ {\color{red}0.4} \end{pmatrix} \end{pmatrix}$$

Quantum: multiple qubits

The Same! $|\phi\rangle \otimes |\psi\rangle$, $|0\rangle|0\rangle = |00\rangle$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
○○

Applications/TD
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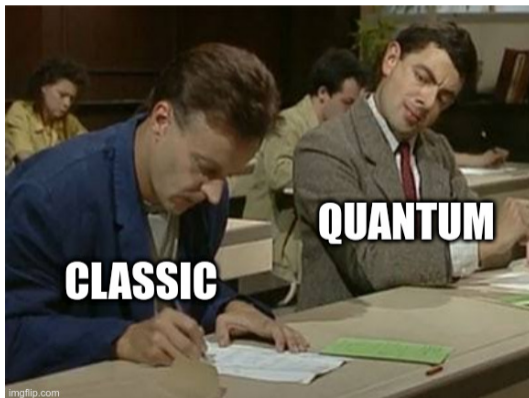
$$|0\rangle \otimes |0\rangle = |00\rangle$$

Tensor product (Kronecker product)

Definition tensor product (Kronecker product)

Same pattern applies for any matrix:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n,2} & \dots & a_{n,m} \end{pmatrix}, B = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n,2} & \dots & a_{n,m} \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B & \dots & a_{1,m}B \\ a_{2,1}B & a_{2,2}B & \dots & a_{2,m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n,2}B & \dots & a_{n,m}B \end{pmatrix}$$

Tensor product (Kronecker product)

Algebra: properties tensor product

- Associativity: $(A \otimes B) \otimes C = A \otimes (B \otimes C)$: omit brackets
- $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$
  Very useful: $(X|\phi\rangle) \otimes (H|\psi\rangle) = (X \otimes H)(|\phi\rangle \otimes |\psi\rangle)$.
- Distributivity: $A \otimes (B + C) = A \otimes B + A \otimes C$, $(A + B) \otimes C = A \otimes C + B \otimes C$.
- $(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha(A \otimes B)$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Tensor product

Exercise + Break time

For any $x \in \{00, 01, 10, 11\}$, compute the vector form of $|x\rangle$ (notation extendable to any encoding of x). For $v_1 = 1/\sqrt{3}|0\rangle + \sqrt{2/3}|1\rangle$ and $v_2 = \begin{pmatrix} \sqrt{2/5} \\ \sqrt{3/5} \end{pmatrix}$, compute[a] $v_1 \otimes v_2$ **using tensor algebra**.

[a]**Poll**: scalar associated to $|01\rangle$ is A=$\sqrt{3/15}$, B=$\sqrt{4/15}$, C=4/15, D=$\sqrt{3/5}$, E=No idea.

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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Tensor product

Answer: part 1

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Tensor product

Answer: part 1

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Tensor product

Answer: part 1

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Tensor product

Answer: part 1

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1 \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Tensor product

Answer: part 1

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Introduction
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Formalisation: single qubit
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Circuit/ZX
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Applications/TD
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Tensor product

Answer: part 2

$$
\begin{aligned}
v_1 \otimes v_1 &= (1/\sqrt{3}|0\rangle + \sqrt{2/3}|1\rangle) \otimes (\sqrt{2/5}|0\rangle + \sqrt{3/5}|1\rangle) \\
&= (1/\sqrt{3}|0\rangle) \otimes (\sqrt{2/5}|0\rangle) + (1/\sqrt{3}|0\rangle) \otimes (\sqrt{3/5}|1\rangle) \\
&+ (\sqrt{2/3}|1\rangle) \otimes (\sqrt{2/5}|0\rangle) + (\sqrt{2/3}|1\rangle) \otimes (\sqrt{3/5}|1\rangle) \\
&= \sqrt{2/15}|0\rangle \otimes |0\rangle + \sqrt{3/15}|0\rangle \otimes |1\rangle \\
&+ \sqrt{4/15}|1\rangle) \otimes |0\rangle + \sqrt{6/15}|1\rangle \otimes |1\rangle \\
&= \sqrt{2/15}|00\rangle + \sqrt{3/15}|01\rangle + \sqrt{4/15}|10\rangle + \sqrt{6/15}|11\rangle = \begin{pmatrix} \sqrt{2/15} \\ \sqrt{3/15} \\ \sqrt{4/15} \\ \sqrt{6/15} \end{pmatrix}
\end{aligned}
$$

Entangled qubits

Correlations = Entangled qubits

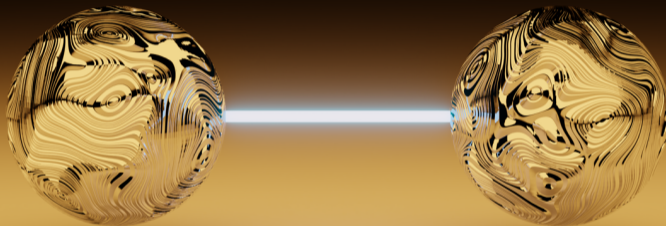Consider: $\Pr[\text{outcome} = 00] = \frac{1}{2}$, $\Pr[\text{outcome} = 11] = \frac{1}{2}$.

Classically: $v = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}$, Quantumly: $v = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Correlations $\implies$ Impossible decompose v as $|\psi\rangle \otimes |\phi\rangle$. State is **entangled**.

Tensor = independent states/operations.

Proof

If $v = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$, then $ad = 0$ and $ac \neq 0$ so $d = 0$. Impossible: $bd \neq 0$.

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Formalisation: single qubit
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Multiple qubits
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Applications/TD
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Multi-qubit gates

Gates

Always unitaries gates $U^\dagger U = I$

- independent matrices on each qubit: tensor, like

$$X \otimes I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- Multi-qubit gates. E.g.: Control-Not (CNOT): $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$\text{CNOT} \left|00\right\rangle = \left|00\right\rangle, \text{CNOT} \left|01\right\rangle = \left|01\right\rangle, \text{CNOT} \left|10\right\rangle = \left|11\right\rangle, \text{CNOT} \left|11\right\rangle = \left|10\right\rangle$

Interest of Dirac notation

Dirac notation vs Matrices

$$|000000\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Bigg\} \; 2^6 - 1 = 63 \text{ zeros}$$

$$\sum_{x=0}^{2^n-1} |x\rangle = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \Bigg\} \; 2^n \text{ zeros}$$

(drop normalization)

Can represent matrices of size 2^n elements concisely.

Dimension of n qubits $= 2^n$ (exponential).

Matrix Notation · Dirac Notation

Interest of Dirac notation

Dirac notation in matrices

Using projectors, we can represent any matrix in Dirac notation. E.g.:

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ gives $X = |1\rangle\langle 0| + |0\rangle\langle 1|$.

Exemple:

$$X|0\rangle = (|1\rangle\langle 0| + |0\rangle\langle 1|)|0\rangle$$
$$= |1\rangle\langle 0|0\rangle + |0\rangle\langle 1|0\rangle$$
$$= |1\rangle \times 1 + |0\rangle \times 0$$
$$= |1\rangle$$

Interest of Dirac notation

Dirac notation and measurement

Measure in the computational basis: for any bit string $s \in \{0,1\}^n$ and any n-qubit state $|\phi\rangle \in \mathcal{H}_{2^n}$:

$$\Pr[\text{outcome} = s] = |\langle s|\phi\rangle|^2$$

State after measurement: $|s\rangle$

Measurement arbitrary basis

If $B = (|\phi_1\rangle, \ldots, |\phi_n\rangle)$ is an orthonormal basis of $\mathcal{H}_n$, we can measure a state $|\psi\rangle$ in basis B:

- $\forall i, \Pr[\text{outcome} = i] = |\langle\phi_i|\psi\rangle|^2$
  New state is $|\phi_i\rangle$

Interest of Dirac notation

Exercice

Compute[a] (using the Dirac notation) the resulting state when applying
$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ on the first qubit of a Bell pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

---

[a]**Poll**: How many positive entries: A=0, B=2, C=3, D=4, E=No idea.

Interest of Dirac notation

Answer: the correct answer is $C = 3$

$$|\psi\rangle = (H \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}((H|0\rangle)|0\rangle + (H|1\rangle)|1\rangle)$$

Then, easy[a] to see that $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$ and $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$.

$$|\psi\rangle = (H \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{2}((|0\rangle + |1\rangle)|0\rangle + (|0\rangle - |1\rangle)|1\rangle)$$

$$= \boxed{\frac{1}{2}(|00\rangle + |10\rangle + |01\rangle - |11\rangle)}$$

---

[a] Either via the matrix form, or by saying that $H = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|)$ and then $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle\langle 0|0\rangle + |1\rangle\langle 0|0\rangle + |0\rangle\langle\cancel{1|0}\rangle - |1\rangle\langle\cancel{1|0}\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Partial measurement

Partial measurement

Multiple qubits, measure **one** qubit i.

E.g.: $|\psi\rangle = \frac{1}{2}|00\rangle - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$, measure first qubit.

- To obtain probability outcome 0: keep only terms with 0 at position i, compute squared norm.
  E.g.: $\frac{1}{2}|00\rangle - \frac{i}{2}|\cancel{10}\rangle + \frac{1}{\sqrt{2}}|\cancel{11}\rangle = \frac{1}{2}|00\rangle$.

  Squared norm: $(\frac{1}{2}\langle 00|)(\frac{1}{2}|00\rangle) = \frac{1}{4}\langle 00|00\rangle = \boxed{\frac{1}{4} = \Pr[\text{outcome} = 0]}$

- To obtain state after measurement: normalize state obtained at last step. E.g.:

$$\frac{\frac{1}{2}|00\rangle}{\|\frac{1}{2}|00\rangle\|_2} = \frac{\frac{1}{2}|00\rangle}{\sqrt{1/4}} = |00\rangle$$

Partial measurement

Exercise

Same idea for outcome 1 and more than 2 qubits: considering
$|\psi\rangle = \frac{i}{\sqrt{2}}|00\rangle - \frac{i}{2}|10\rangle + \frac{1}{2}|11\rangle$ what[a] is the probability of measuring 1, and what is the final state?

---
    [a]**Poll**: A$=1/\sqrt{2}$, B$=1/4$, C$=$Other, D$=$No idea.

Partial measurement

Answer: the correct answer is C

$\frac{i}{\sqrt{2}}|00\rangle - \frac{i}{2}|10\rangle + \frac{1}{2}|11\rangle$

Squared norm:

$$\left(\frac{i}{2}\langle10| + \frac{1}{2}\langle11|\right)\left(\frac{-i}{2}|10\rangle + \frac{1}{2}|11\rangle\right) = \frac{1}{4}\langle10|10\rangle + \frac{i}{4}\langle10|11\rangle + \frac{-i}{4}\langle11|10\rangle + \frac{1}{4}\langle11|11\rangle$$

$$= \frac{1}{4} + \frac{1}{4} = \boxed{\frac{1}{2} = \Pr[\text{outcome } = 1]}$$

New state:

$$\frac{\frac{-i}{2}|10\rangle + \frac{1}{2}|11\rangle}{\sqrt{1/2}} = \boxed{\frac{-i}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle}$$
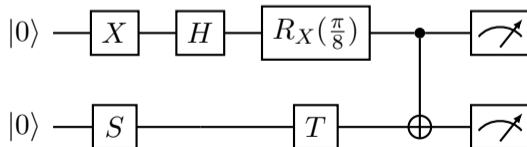
Circuit model

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Circuit model

Universal set of gates

Time: left to right. Clifford+T=Universal:

- Clifford: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $S = R_Z(\frac{\pi}{2}) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, CNOT

- $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$

Circuit model

Universal set of gates

Time: left to right. Clifford+T=Universal:

- Clifford: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $S = R_Z(\frac{\pi}{2}) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, CNOT

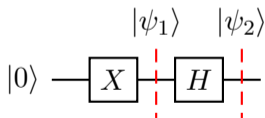- $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$

$$|0\rangle - \boxed{X} -$$
$$\qquad\qquad = (X \otimes S)|00\rangle = (X|0\rangle) \otimes (S|0\rangle)$$
$$|0\rangle - \boxed{S} -$$

Circuit model

Universal set of gates

Time: left to right. Clifford+T=Universal:

- Clifford: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $S = R_Z(\frac{\pi}{2}) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, CNOT

- $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$

$$|\psi_1\rangle \quad |\psi_2\rangle$$

$$|0\rangle - \boxed{X} - \boxed{H} -$$

Exercice

Poll: Final state is: A=$|0\rangle$, B=$|1\rangle$, C=$|+\rangle$, D=$|-\rangle$, E=No idea.

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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Circuit model

Universal set of gates

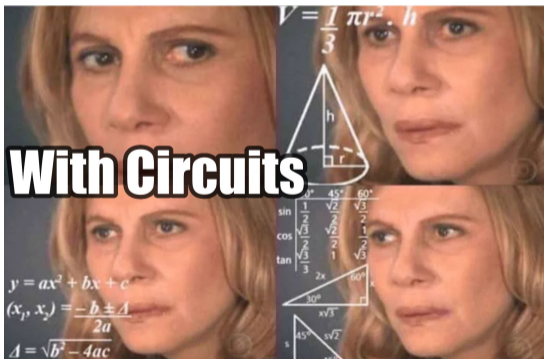Time: left to right. Clifford+T=Universal:

- Clifford: $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $S = R_Z(\frac{\pi}{2}) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, CNOT

- $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$

$$|\psi_1\rangle \quad |\psi_2\rangle$$

$$|0\rangle \longrightarrow \boxed{X} \dashv \boxed{H} \dashv$$

$$|\psi_1\rangle = X|0\rangle = |1\rangle$$
$$|\psi_2\rangle = H|\psi_1\rangle = H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$$

ZX-Calculus

ZX-Calculus

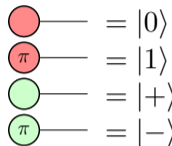### This presentation

Mini-introduction to ZX-Calculus: for intuition only (maybe more: talk Ross Duncan)
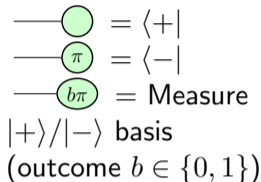$\Rightarrow$ Bonus!

Idea

Each graph represents a matrix. Computation $=$ graph rewriting.

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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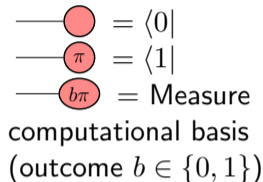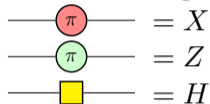
ZX-Calculus

(Subset of) Rules (same inverted colors):

- Only topology matters
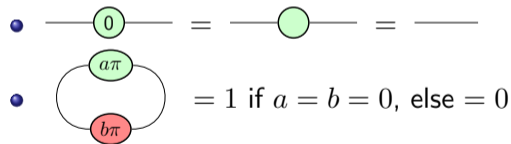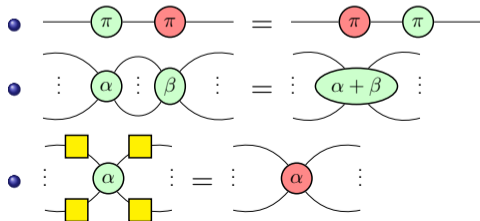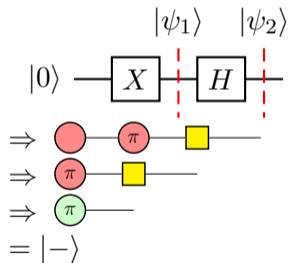
Bell pair: CNOT:

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Formalisation: single qubit
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Multiple qubits
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ZX-Calculus

Find circuit to produce a Bell pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Check (circuit and/or ZX) correctness.

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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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ZX-Calculus

$$|\psi_1\rangle \quad |\psi_2\rangle$$

$$|0\rangle \; - \boxed{X} \; - \boxed{H} \; -$$

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$\Rightarrow$ 🔴π — 🟨

$\Rightarrow$ 🟢π

$= |-\rangle$

Find circuit to produce a Bell pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Check (circuit and/or ZX) correctness.

$$|0\rangle \; - \boxed{H} \; - \bullet \; -$$

$$|0\rangle \; - \oplus \; -$$

Step1:
$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
Step 2:

$$\mathrm{CNOT}(H|0\rangle) \otimes |0\rangle$$

$$= \mathrm{CNOT} \, \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

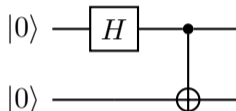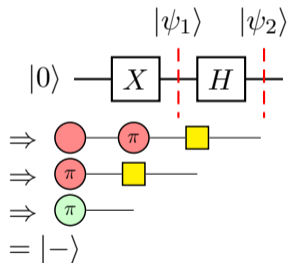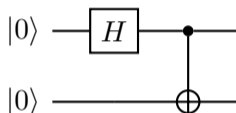## ZX-Calculus

Find circuit to produce a Bell pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Check (circuit and/or ZX) correctness.

Step1:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Step 2:

$$\mathrm{CNOT}(H|0\rangle) \otimes |0\rangle$$

$$= \mathrm{CNOT}\, \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Superdense coding: Goal: transmit 2 bits a and b by sending a single bit or qubit.

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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Superdense coding: Goal: transmit 2 bits a and b by sending a single bit or qubit. Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Superdense coding: Goal: transmit 2 bits a and b by sending a single bit or qubit. Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Superdense coding: Goal: transmit 2 bits a and b by sending a single bit or qubit. Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

(a, b)

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Superdense coding: Goal: transmit 2 bits a and b by sending a single bit or qubit. Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Superdense coding: Goal: transmit 2 bits a and b by sending a single bit or qubit. Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Superdense coding: Goal: transmit 2 bits a and b by sending a single bit or qubit. Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

(a, b)

Introduction
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Formalisation: single qubit
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Multiple qubits
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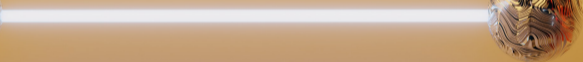
Circuit/ZX
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Applications/TD
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Superdense coding: Goal: transmit 2 bits a and b by sending a single bit or qubit.
Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

(a, b)

Introduction
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Formalisation: single qubit
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Multiple qubits
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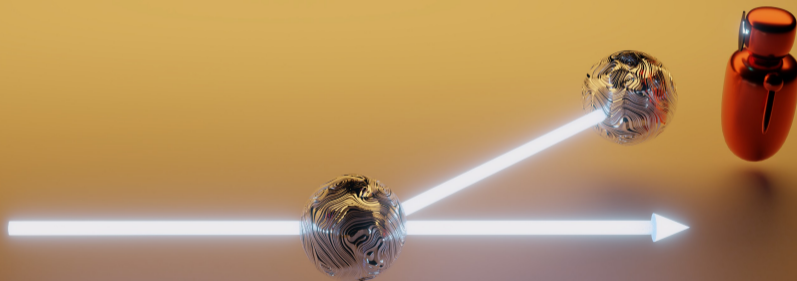
Circuit/ZX
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Applications/TD
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Superdense coding: Goal: transmit 2 bits a and b by sending a single bit or qubit. Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

(a, b)

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Superdense coding: Goal: transmit 2 bits a and b by sending a single bit or qubit.
Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)
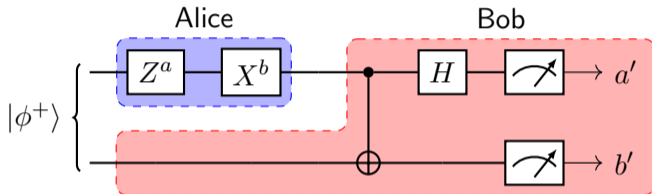
Superdense coding

> ### Exercice
>
> – Level 1: Check for all $(a, b) \in \{0, 1\}^2$, $(a, b) = (a', b')$.
> – Level 2: Do it "in one-go" by keeping the variables
> – Level 3: Do it with ZX-Calculus.

Superdense coding

Correction (level 1: check for all (a, b))

(Thanks Watrous 🙏)

ab	state after step 1	state after step 2	state after step 4	state after step 5
00	$\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$	$\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$	$\left(\frac{1}{\sqrt{2}}\|0\rangle + \frac{1}{\sqrt{2}}\|1\rangle\right)\|0\rangle$	$\|00\rangle$
01	$\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$	$\frac{1}{\sqrt{2}}\|10\rangle + \frac{1}{\sqrt{2}}\|01\rangle$	$\left(\frac{1}{\sqrt{2}}\|1\rangle + \frac{1}{\sqrt{2}}\|0\rangle\right)\|1\rangle$	$\|01\rangle$
10	$\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$	$\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$	$\left(\frac{1}{\sqrt{2}}\|0\rangle - \frac{1}{\sqrt{2}}\|1\rangle\right)\|0\rangle$	$\|10\rangle$
11	$\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$	$\frac{1}{\sqrt{2}}\|10\rangle - \frac{1}{\sqrt{2}}\|01\rangle$	$\left(\frac{1}{\sqrt{2}}\|1\rangle - \frac{1}{\sqrt{2}}\|0\rangle\right)\|1\rangle$	$-\|11\rangle$

Superdense coding

Correction (level 2: variables)

– Starting with state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

– Apply Z_1^a: $\frac{1}{\sqrt{2}}(|00\rangle + (-1)^a|11\rangle)$

– Apply X_1^b: $\frac{1}{\sqrt{2}}(|b0\rangle + (-1)^a|(1 \oplus b)1\rangle)$

– Apply $\text{CNOT}_{1,2}$: (use $\forall(a,b) \in \{0,1\}^2$, $\text{CNOT}\,|a\rangle|b\rangle = \text{CNOT}\,|a\rangle|b \oplus a\rangle$)

$\frac{1}{\sqrt{2}}(|bb\rangle + (-1)^a|(1 \oplus b)(1 \oplus (b \oplus 1))\rangle) = \frac{1}{\sqrt{2}}(|bb\rangle + (-1)^a|(1 \oplus b)b\rangle) = $

$\frac{1}{\sqrt{2}}(|b\rangle + (-1)^a|(1 \oplus b)\rangle)|b\rangle = \frac{1}{\sqrt{2}}(-1)^{ab}(|0\rangle + (-1)^a|1\rangle)|b\rangle$

– Apply H_1: $(-1)^{ab}|a\rangle|b\rangle$

– Measure: a,b.

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Superdense coding

Correction (level 3: ZX)
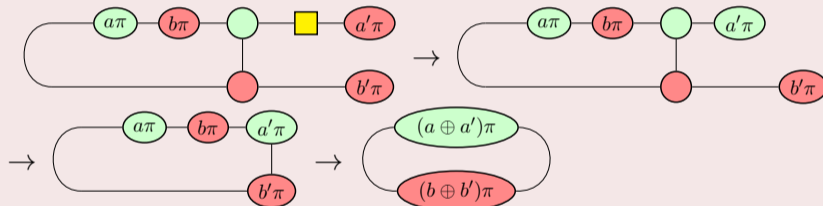
If $a \oplus a' = 0$ and $b \oplus b' = 0$, then the above graph equals 1 (i.e. proba of measuring $a' = a$ and $b' = b$ is 1) otherwise the proba is 0: so $a' = a$ and $b' = b$.

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Quantum Teleportation: Goal: transmit a qubit using a purely classical channel.

Introduction
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Formalisation: single qubit
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Multiple qubits
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Quantum Teleportation: Goal: transmit a qubit using a purely classical channel.

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Quantum Teleportation: Goal: transmit a qubit using a purely classical channel.
Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

Introduction
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Formalisation: single qubit
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Multiple qubits
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Applications/TD
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Quantum Teleportation: Goal: transmit a qubit using a purely classical channel.
Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
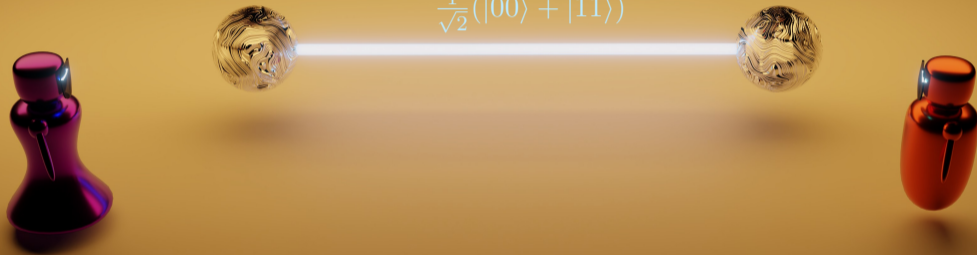○○

Applications/TD
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Quantum Teleportation: Goal: transmit a qubit using a purely classical channel.
Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Quantum Teleportation: Goal: transmit a qubit using a purely classical channel.
Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Quantum Teleportation: Goal: transmit a qubit using a purely classical channel.
Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

Introduction
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Formalisation: single qubit
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Multiple qubits
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Quantum Teleportation: Goal: transmit a qubit using a purely classical channel.
Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Quantum Teleportation: Goal: transmit a qubit using a purely classical channel. Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Quantum Teleportation: Goal: transmit a qubit using a purely classical channel.
Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

(a, b)

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Quantum Teleportation: Goal: transmit a qubit using a purely classical channel.
Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

X^b

(a, b)

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Quantum Teleportation: Goal: transmit a qubit using a purely classical channel.
Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

$$\mathcal{Z}^a$$

$$(a, b)$$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Quantum Teleportation: Goal: transmit a qubit using a purely classical channel.
Impossible, except with **pre-shared** quantum resources (Bell pair, a.k.a. EPR pair)

(a, b)

Teleportation
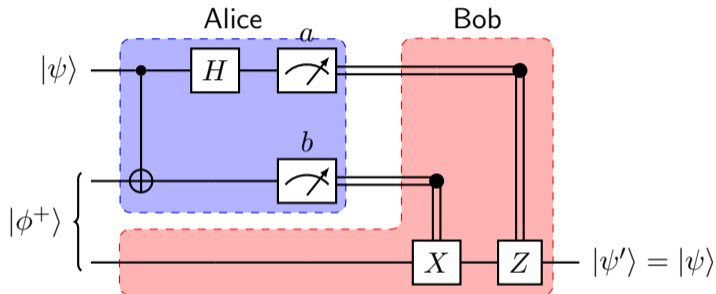
Exercice

– Level 1: Check that for all $(a,b) \in \{0,1\}^2$, $|\psi'\rangle = |\psi\rangle$.
– Level 2: Do it with ZX-Calculus.
– Level 3: Do not look at the circuit: recover it directly using ZX-Calculus.

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Teleportation

Answer (level 1)

– Step 0: $(\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

Teleportation

Answer (level 1)

– Step 0: $(\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

– Step 1 $\mathrm{CNOT}_{1,2}$: $\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$

Teleportation

Answer (level 1)

– Step 0: $(\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

– Step 1 $\mathrm{CNOT}_{1,2}$: $\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$

– Step 2 H (reminder: $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$):

$\frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle)$

$= \frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle) + \frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) + \frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle)$

Teleportation

Answer (level 1)

– Step 0: $(\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

– Step 1 $\mathrm{CNOT}_{1,2}$: $\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$

– Step 2 H (reminder: $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$):

$\frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle)$

$= \frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle) + \frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) + \frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle)$

– Step 3 Measure: Case 1 = outcome 00 ($a = 0$, $b = 0$).
Post-measured state: $\alpha|0\rangle + \beta|1\rangle$
No need to apply any X/Z: $|\phi'\rangle = |\phi\rangle$. Ok!

Teleportation

Answer (level 1)

– Step 0: $(\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

– Step 1 $\mathrm{CNOT}_{1,2}$: $\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$

– Step 2 H (reminder: $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$):

$$\frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle)$$

$$= \frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle) + \frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) + \frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle)$$

– Step 3 Measure: Case 2 = outcome 01 ($a = 0$, $b = 1$).

Post-measured state: $\alpha|1\rangle + \beta|0\rangle$

After applying X: $|\phi'\rangle = \alpha|0\rangle + \beta|1\rangle = |\phi\rangle$. Ok!

Teleportation

Answer (level 1)

– Step 0: $(\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

– Step 1 $\mathrm{CNOT}_{1,2}$: $\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$

– Step 2 H (reminder: $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$):

$\frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle)$

$= \frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle) + \frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) + \frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle)$

– Step 3 Measure: Case 3 = outcome 10 ($a = 1$, $b = 0$).

Post-measured state: $\alpha|0\rangle - \beta|1\rangle$

After applying Z: $|\phi'\rangle = \alpha|0\rangle + \beta|1\rangle = |\phi\rangle$. Ok!
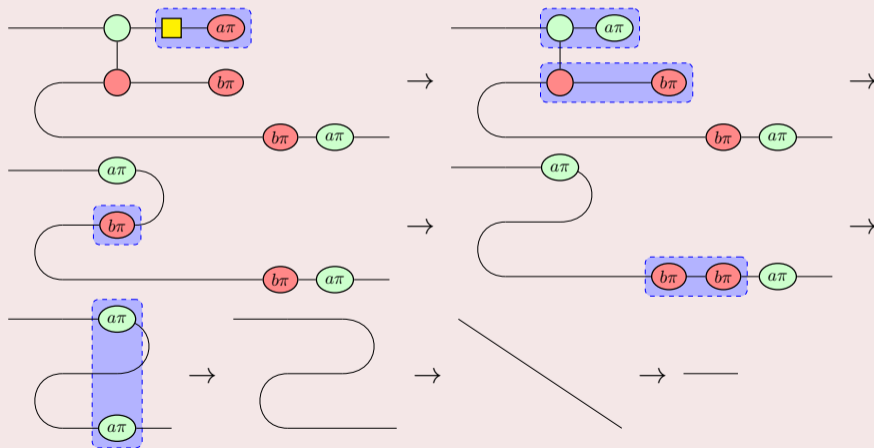
Teleportation

Answer (level 1)

– Step 0: $(\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

– Step 1 $\text{CNOT}_{1,2}$: $\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$

– Step 2 H (reminder: $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$):

$\frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle)$

$= \frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle) + \frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) + \frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle)$

– Step 3 Measure: Case 4 = outcome 11 ($a = 1$, $b = 1$).

Post-measured state: $\alpha|1\rangle - \beta|0\rangle$

After applying X: $\alpha|0\rangle - \beta|1\rangle = |\phi\rangle$. After applying Z: $\alpha|0\rangle + \beta|1\rangle = |\phi\rangle$ .Ok!
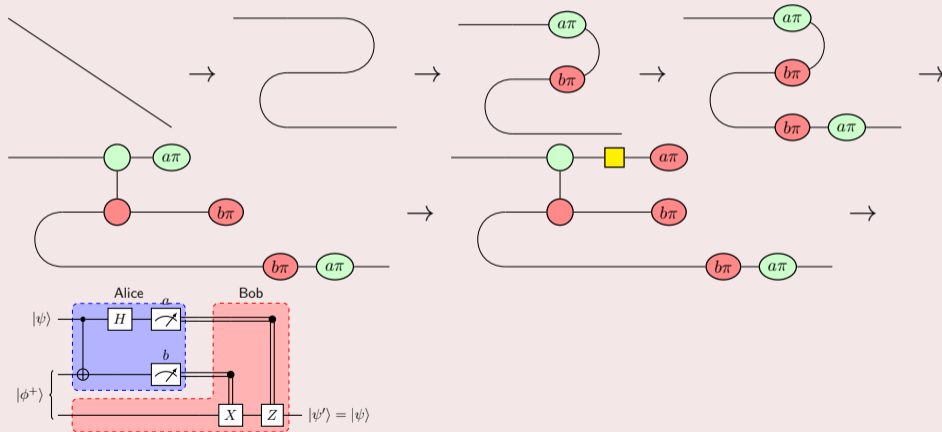
Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Teleportation

Answer: level 2

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Teleportation

Answer: level 3

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Deutsch's Algorithm

Deutsch's Quantum Algorithm

Goal: given a function $f \colon \{0,1\} \to \{0,1\}$ (as black-box), check if:

- f is constant
- f is balanced ($|f^{-1}(0)| = |f^{-1}(1)|$)

| Input	f_0	f_1	f_2	f_3
0	0	0	1	1
1	0	1	0	1

Exercice

Which functions are constant? Balanced?
Poll: f_1 is A=Constant, B=Balanced, C=No idea.

Deutsch's Algorithm

Deutsch's Quantum Algorithm

Goal: given a function $f \colon \{0,1\} \to \{0,1\}$ (as black-box), check if:

- f is constant
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Input	f_0	f_1	f_2	f_3
0	0	0	1	1
1	0	1	0	1

Exercice

Which functions are constant? Balanced?

Poll: f_1 is A=Constant, B=Balanced, C=No idea.$\Rightarrow$ Answer B

Deutsch's Algorithm

Black-Box

Classically:

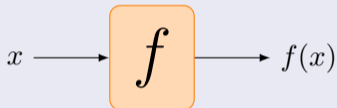$$x \longrightarrow \boxed{f} \longrightarrow f(x)$$

Quantumly:

$$|x\rangle \longrightarrow \boxed{f} \longrightarrow |f(x)\rangle$$
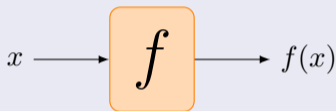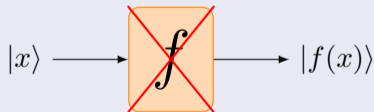
Problem: not unitary. For example $f_0(x) = 0$ would give $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Deutsch's Algorithm

Black-Box

Classically:

$$x \longrightarrow \boxed{f} \longrightarrow f(x)$$

Quantumly (can be extended to any f):

$$|x\rangle \longrightarrow \boxed{U_f} \longrightarrow |x\rangle$$
$$|0\rangle \longrightarrow \qquad\quad \longrightarrow |f(x)\rangle$$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Deutsch's Algorithm

Black-Box

Classically:

$$x \longrightarrow \boxed{f} \longrightarrow f(x)$$
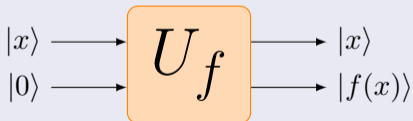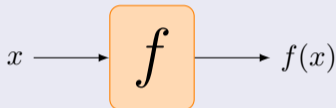
Quantumly (can be extended to any f, $\oplus$ = bitwise XOR):
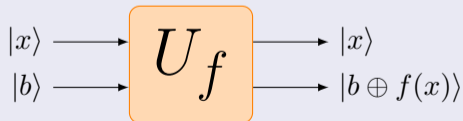
$$|x\rangle \longrightarrow \boxed{U_f} \longrightarrow |x\rangle$$
$$|b\rangle \longrightarrow \boxed{U_f} \longrightarrow |b \oplus f(x)\rangle$$

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Deutsch's Algorithm

Exercice

Which matrix corresponds to U_{f_1}, where $f_1(x) = x$?
Poll: A=I, B=$\mathrm{CNOT}_{1,2}$, C=$\mathrm{CNOT}_{2,1}$, D=Other, E=No idea.

Deutsch's Algorithm

Exercice

Which matrix corresponds to U_{f_1}, where $f_1(x) = x$?
Poll: A=I, B=$\mathrm{CNOT}_{1,2}$, C=$\mathrm{CNOT}_{2,1}$, D=Other, E=No idea.

Answer: B

Method 1:
$U_{f_1}|00\rangle = |0\rangle|0 \oplus f_1(0)\rangle = |00\rangle$
$U_{f_1}|01\rangle = |0\rangle|1 \oplus f_1(0)\rangle = |01\rangle$
$U_{f_1}|10\rangle = |1\rangle|0 \oplus f_1(1)\rangle = |11\rangle$
$U_{f_1}|11\rangle = |1\rangle|1 \oplus f_1(1)\rangle = |10\rangle$

So:

$$U_{f_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \mathrm{CNOT}_{1,2}$$

Deutsch's Algorithm

Exercice

Which matrix corresponds to U_{f_1}, where $f_1(x) = x$?
Poll: A$=I$, B$=$CNOT$_{1,2}$, C$=$CNOT$_{2,1}$, D$=$Other, E$=$No idea.

Answer: B

Method 2:
$U_{f_1}(|x\rangle|b\rangle) = |x\rangle|b \oplus f_1(x)\rangle = |x\rangle|b \oplus x\rangle = \mathrm{CNOT}_{1,2}(|x\rangle|b\rangle)$

Introduction
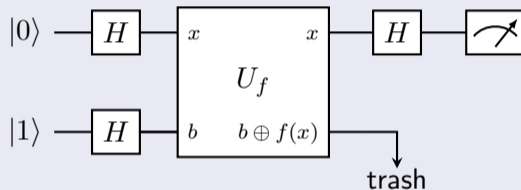○○○○○○○○○○○○○○

Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Deutsch's Algorithm

Deutsch's problem: classically vs quantumly

Classically: **2** calls to know if f is balanced or constant.
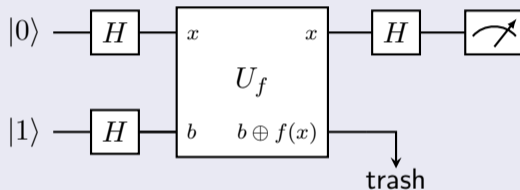Quantumly: **1** calls to know if f is balanced or constant.

Introduction
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Formalisation: single qubit
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Multiple qubits
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Circuit/ZX
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Applications/TD
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Deutsch's Algorithm

Deutsch's algorithm

Deutsch's Algorithm

Deutsch's algorithm

Exercice

Check that measurement is 0 when f is constant, and 1 when f is balanced.

Deutsch's Algorithm

Answer

Initial state: $|0\rangle|1\rangle$

Step 1 (H_1 and H_2): $(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle))(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) = \frac{1}{2}|0\rangle(|0\rangle - |1\rangle) + \frac{1}{2}|1\rangle(|0\rangle - |1\rangle)$

Step 2 (U_f):

$$\frac{1}{2}|0\rangle(|0 \oplus f(0)\rangle - |1 \oplus f(0)\rangle) + \frac{1}{2}|1\rangle(|0 \oplus f(1)\rangle - |1 \oplus f(1)\rangle)$$

Using $\forall a \in \{0, 1\}, |0 \oplus a\rangle - |1 \oplus a\rangle = (-1)^a(|0\rangle - |1\rangle)$:

$$\frac{(-1)^{f(0)}}{2}|0\rangle(|0\rangle - |1\rangle) + \frac{(-1)^{f(1)}}{2}|1\rangle(|0\rangle - |1\rangle) = \frac{(-1)^{f(0)}}{\sqrt{2}}\left(|0\rangle + (-1)^{f(0) \oplus f(1)}|1\rangle\right)\left(\frac{|0\rangle}{\sqrt{2}} - \frac{|1\rangle}{\sqrt{2}}\right)$$

Step 3 (Trash+H): $(-1)^{f_0}|f(0) \oplus f(1)\rangle$. Measure $b := f(0) \oplus f(1)$, and $b = 0$ iff f constant.

Conclusion

The end: I hope you survived (enjoyed?).

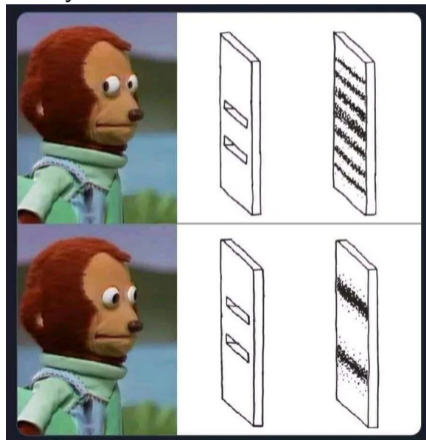http://www.quickmeme.com/meme/3utmn3

Conclusion

Summary: Quantum states can be in superposition.

https://www.justpo.st/channel/schrodinger

Conclusion

Summary: Observation disturbs the state.

https://9gag.com/gag/a9EjQMK

Conclusion

Summary: States can't be copied.

http://www.alibati.com/horse
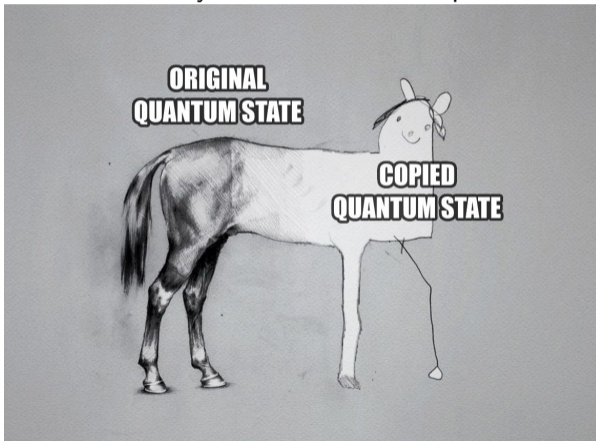
Conclusion

Summary: Quantum can improve speed/security.

Conclusion

Thanks!